PI degree of Quantum Algebras at Roots of Unity

Alex Rogers   Stéphane Launois
University of Kent
ar486@kent.ac.uk

1. Polynomial Identity (PI) algebras

Let R be an algebra over a field, k.

- PI algebra: R satisfies a monic polynomial $f \in \mathbb{Z}[X]$, i.e. $f(r_1, \dots, r_s) = 0 \quad \forall r_i \in R$.
- minimal degree of R: the least degree of all poynomial identites for R.
- PI degree: If R is prime then PI-deg$(R) = \frac{1}{2}$(minimal degree of R).

Examples

1. Quantum affine space $\mathcal{O}_{q^\lambda}(k^N)$ when $q^\ell = 1$. (λ is matrix of commutation relations.)
   – When $N = 2$ we get the quantum affine plane $k_q[X, Y]$ where $XY = qYX$.
2. Uni-parameter quantum matrices $\mathcal{O}_q(M_{m,n}(k))$ when $q^\ell = 1$.
3. Multiparameter quantum matrices $\mathcal{O}_{\lambda,\mathbf{p}}(M_{m,n}(k))$ when λ and $p_{i,j}$ are roots of unity.

Facts about PI degree when R is prime

– PI-degree doesn't change under localisation: PI-deg(RS^{-1}) =PI-deg(R).

– If k algebraically closed and R affine (as in all the examples above) then PI-degree gives an upper bound on the dimension of the irreducible representations of R.

Theorem 1 ([3]). *If $\mathcal{O}_{q^\lambda}(k^N)$ is a quantum affine space with $\boldsymbol{\lambda} = (\lambda_{ij})_{i,j}$ and q a primitive ℓ^{th} root of unity. Then PI-deg$(\mathcal{O}_{q^\lambda}(k^N)) = \sqrt{h}$ where h is the cardinality of the image of*

$$\pi \circ \boldsymbol{\lambda} : \mathbb{Z}^N \longrightarrow \mathbb{Z}^N \longrightarrow (\mathbb{Z}/\ell\mathbb{Z})^N.$$

Theorem 2 ([4] & [6]). *Given a **suitable** iterated Ore extension $R = k[X_1] \dots [X_N; \sigma_N, \delta_N]$ with automorphisms $\sigma_i(X_j) = q^{\lambda_{ij}} X_j X_i$ and q a primitive ℓ^{th} root of unity. Then R is a PI ring and* Frac$(R) \cong$ Frac$(\mathcal{O}_{q^\lambda}(k^N))$. *Therefore PI-deg$(R)$ =PI-deg$(\mathcal{O}_{q^\lambda}(k^N))$.*

Objective: Compute PI-deg(R/P) for **suitable** iterated Ore extensions R and completely prime ideals $P \lhd R$.

Strategy: Extend methods in [4] and [2] to use localisations to get a quantum affine space R' such that Frac(R/P) =Frac(R') and then apply Theorem 1 to R'.

2. Calculating the cardinality of $\mathrm{Im}(\pi \circ \lambda)$

Since $\boldsymbol{\lambda}$ is a skew-symmetric, integral matrix then it has a congruent skew-normal form:

$$U\boldsymbol{\lambda}U^T = S = \begin{pmatrix} \begin{smallmatrix} 0 & h_1 \\ -h_1 & 0 \end{smallmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{smallmatrix} 0 & h_s \\ -h_s & 0 \end{smallmatrix} \\ & & & 0 \end{pmatrix} \in M_N(\mathbb{Z})$$

The $h_i \in \mathbb{Z}^*$ are the invariant factors of λ with the property $h_i \mid h_j$ for all $i < j$.

[7, Lemma 2.4] $\Rightarrow$ The quantum tori associated to $\boldsymbol{\lambda}$ and S are isomorphic, hence

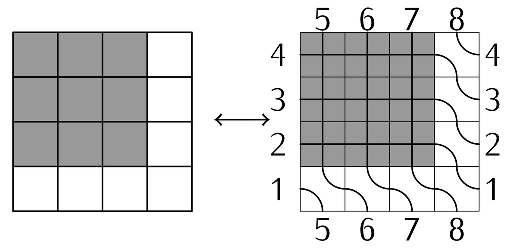
$$\text{PI-deg}(\mathcal{O}_{q^\lambda}(k^N)) = \text{PI-deg}(\mathcal{O}_{q^S}(k^N)).$$

$\therefore$ We can replace $\boldsymbol{\lambda}$ in Theorem 1 with S and see that $h = \mathrm{card}(\mathrm{Im}(\pi \circ S))$.

The shape of S makes it clear that *h depends on the dimension of $ker(\boldsymbol{\lambda})$ and the values of its invariant factors h_i*. Cauchon diagrams can help here.

3. Cauchon Diagrams

- Diagram, D: an $m \times n$ grid filled with black and white squares.
- Cauchon diagram, C: For any black square, either all squares strictly above it or all squares strictly to the left of it are black.
- Matrix associated to D, $M(D)$: To each diagram D with N white squares we can form an $N \times N$ skew-symmetric, integral matrix, $M(D)$.
- Pipe dreams: Label sides of diagram (as shown) and lay pipes over the squares - place a "cross" over black squares and a "hyperbola" over white squares.
- Toric permutation of D, τ: Read off the toric permutation τ by defining $\tau(i)$ to be the label reached (on the left or top of D) by following the path starting at i (on the right or bottom of D):



$\Rightarrow \quad \tau = (12348765)$

Proposition 1 ([1]). *Let D be a diagram with restricted permutation τ. Then the dimension of $\ker(M(D))$ is the number of odd cycles (even length) in the disjoint cycle decomposition of τ.*

Proposition 2. *Let $M(C)$ be the matrix associated to a Cauchon diagram C. Then all invariant factors of $M(C)$ are powers of 2.*

$\therefore$ Given a specific Cauchon diagram C we can compute the PI degree of its associated quantum affine space, $\mathcal{O}_{q^{M(C)}}(k^N)$ when $q^\ell = 1$ and ℓ is odd.

4. Quantum Determinantal Rings

Theorem 3. *Let $R_t := \mathcal{O}_q(M_n(k))/I_t$ where I_t is the two-sided ideal of $R := \mathcal{O}_q(M_n(k))$ generated by all $(t+1) \times (t+1)$ quantum minors and $q \in k^*$ is a primitive ℓ^{th} root of unity with ℓ odd. Then PI-deg$(R_t) = \ell^{\frac{2nt - t^2 - t}{2}}$.*

- We have actually computed irreducible representations for R_t of correct dimension.

Sketch proof of Theorem 3:

– [5, Lemma 4.4]: For a $t \times t$ quantum minor $\delta \in R$ and it's canonical image $\overline{\delta} \in R_t$,

$$R_t[\overline{\delta}^{-1}] \cong A_t[\overline{\delta^{-1}}]$$

$\therefore$ PI-deg(R_t) =PI-deg(A_t).

– $A_t \subseteq R$ can be written as a **suitable** iterated Ore extension so, applying Theorem 2:

$$\text{PI-deg}(A_t) = \text{PI-deg}(\mathcal{O}_{q^\lambda}(k^N))$$

where $N = 2nt - t^2$ is the number of generators in A_t and $\boldsymbol{\lambda}$ is known.

– $\boldsymbol{\lambda} = M(C)$ where C is the $n \times n$ Cauchon diagram with the last t columns and rows of squares left white so all invariant factors, h_i, are powers of 2 (by Proposition 2).

– Using Proposition 1 we proved that the dimension of $\ker(M(C))$ is t. Conclude using Theorem 1:

$$\text{PI-deg}(R_t) = \text{PI-deg}(\mathcal{O}_{q^\lambda}(k^N)) = \ell^{\frac{N-t}{2}}. \quad \square$$

5. Deleting Derivations Algorithm

Aim: Extend Theorem 2 to incoorporate quotient algebras R/P by adapting methods in [2] to work at roots of unity.

Take iterated Ore extensions satisfying **suitable** criteria (as in Theorem 2):

$$R := k[X_1][X_2; \sigma_2, \delta_2] \cdots [X_N; \sigma_N, \delta_N].$$

Note that examples 1-3 are all prime PI algebras satisfying these criteria.

Let

$$R' := \mathcal{O}_{q^\lambda}(k^N) = k[T_1][T_2; \sigma_2] \cdots [T_N; \sigma_N].$$

Then Theorem 2 $\Rightarrow$ Frac$(R) \cong$ Frac(R'). We extend this result to quotients:

- Canonical embedding: Denote the set of completeley prime ideals in R as C.Spec(R). Then:
  $$\psi : \text{C.Spec}(R) \hookrightarrow \text{C.Spec}(R')$$
- For $P \in$ C.Spec(R) we have Frac(R/P) = Frac$(R'/\psi(P))$.
- Let $W = \mathscr{P}(\{1, \dots, N\})$, the power set of $\{1, \dots, N\}$, and define $J_w := \langle T_i \mid i \in w \rangle \in$ C.Spec(R') for some $w \in W$. If $\psi(P) = J_w$ then
  $$\text{Frac}(R/P) = \text{Frac}(R'/J_w)$$
  where R'/J_w is a quantum affine space.

Question: When is $J_w \in \mathrm{Im}(\psi)$?
Answer: Unclear in general but for quantum matrices this is related to Cauchon diagrams.

Proposition 3. *Let $R = \mathcal{O}_{\lambda,\mathbf{p}}(M_{m,n}(k))$ (where λ and all $p_{i,j}$ are roots of unity) and $W = \mathscr{P}(\{(1,1), (1,2), \dots, (m,n)\})$. Then*

$$J_w \in \mathrm{Im}(\psi) \iff \mathcal{C}_w \text{ is Cauchon}$$

where $\mathcal{C}_w$ is the $m \times n$ diagram whose square in position (i,j) is coloured black if and only if $(i,j) \in w$.

$\rightarrow$ Methods above give ways to calculate PI-deg(R'/J_w) and hence PI-deg$(R/\psi^{-1}(J_w))$.

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